

Research Article

COUNTING OF DISJOINT HAMILTON CYCLES IN A UNITARY DIVISOR CAYLEY GRAPH

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Abstract

Let $n \geq 1$ be an integer and let S be the set of unitary divisor of n . Then the set $S^* = \{s, n - s \mid s \in S\}$ is a symmetric subset of the group $(\mathbb{Z}_n, \oplus)$, the additive abelian group of integers modulo n . The Cayley graph of $(\mathbb{Z}_n, \oplus)$ associated with the above symmetric subset S^* is called the **unitary divisor Cayley graph** and it is denoted by $G(\mathbb{Z}_n, U_n)$. That is, the graph $G(\mathbb{Z}_n, U_n)$ is the graph whose vertex set is $V = \{0, 1, 2, \dots, n - 1\}$ and the edge set E is the set of all ordered pairs of vertices x, y such that either $x - y \in S^*$, or, $y - x \in S^*$. In this Paper we studied counting of disjoint Hamilton cycles in a unitary divisor Cayley graph.

Keywords: Unitary Divisor Function, Cayley graph, unitary divisor Cayley graph, bipartite graph, Hamilton cycle, Eulerian graph.

1. Introduction

For standard terminology and notation in graph theory we refer Bondy and Murty [4] and Harary [5] and for number theoretic notations Apostol [2]. Nathanson [7] was the pioneer in introducing the concepts of number theory and thus paved way for the study of a new class of graphs called Arithmetic Graphs arising by defining adjacency using various arithmetic functions. The theory of groups provides an interesting and powerful abstract approach to the study of symmetries of various graphs. A new class of graphs namely, Cayley Graphs can be constructed by making use of a group (X, S) and a symmetric subset S of X (a subset S of X is called a symmetric subset if $s \in S \Rightarrow s^{-1} \in S$). It is the graph $G(X, S)$ whose vertex set is X and edge set $\{(x, y) \mid x, y \in X \text{ and either } xy^{-1} \in S \text{ or } yx^{-1} \in S\}$. The degree of a vertex v , $d(v)$ in G is the number of edges incident with a vertex v . If degree of each vertex is equal, say r in G , then G is called r -regular graph. A graph is a complete graph if every vertex adjacent to all other vertices of the graph. A walk in a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, \dots, e_n, v_n$, beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. It is closed if $v_0 = v_n$. A closed walk in which all the edges are distinct is called circuit. An Eulerian circuit in a graph G is a circuit containing every edge of G and G is an Eulerian graph if it contains an Eulerian circuit. A Hamiltonian cycle in a graph G is a cycle containing every vertex of G and G is called a Hamiltonian graph if it contains a Hamiltonian cycle. The cycle $(0 \ 1 \ 2 \ \dots \ n - 1)$ is called the outer Hamilton Cycle. A bipartite graph is a graph whose vertex set can be partitioned into two disjoint subsets X and Y (that is $V(G) = X \cup Y, X \cap Y = \emptyset$) such that each edge has one end in X and other end in Y .

2. Unitary Divisor Cayley Graph and Properties

Let $n \geq 1$ be an integer. Consider the set $\mathbb{Z}_n = \{0, 1, \dots, \bar{n} \equiv \bar{1}\}$ of residue classes modulo n . Since $\bar{n} = \bar{0}$, we can as well denote $\mathbb{Z}_n = \{1, 2, \dots, \bar{n}\}$. In view of this, the set $\mathbb{Z}_n$ is henceforth represented by

$Z_n = \{1, 2, \dots, n\}$, or, simply $Z_n = \{1, 2, \dots, n\}$. In the abelian group $(Z_n, \oplus)$, n is the identity element and $n - r$ is the inverse of r in $(Z_n, \oplus)$.

Definition 2.1: Let $n \geq 1$ be an integer. A divisor d of n which is such that $(n, \frac{d}{n}) = 1$ is called a **unitary divisor** of n . The number of unitary divisors of n is denoted by $u(n)$ and the set of unitary divisors of n is denoted by U_n .

For example, for $n = 8$, the unitary divisors are 1, 8 and $u(n) = 2$, while for $n = 120$, the unitary divisors are 1, 3, 5, 8, 15, 24, 40, 120 and $u(n) = 8$.

In the following table the unitary divisors and their number are given for integers n up to $n = 15$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Unitary divisors of n	1	1, 2	1, 3	1, 4	1, 5	1, 2, 3	1, 7	1, 8	1, 9	1, 2, 5, 10	1, 11	1, 3, 4, 12	1, 13	1, 2, 7, 14	1, 3, 5, 15
$u(n)$	1	2	2	2	2	4	2	2	2	4	2	4	2	4	4

Let S be the set of unitary divisors of n , **other than n** . The set S need not be a symmetric subset of the group $(Z_n, \oplus)$. For example, for $n = 12$, the set S of unitary divisors of n , other than n is $S = \{1, 3, 4\}$. Now for $3 \in S$, its inverse in $(Z_{12}, \oplus)$ is

$n - 3 = 12 - 3 = 9$, which is not a unitary divisor of 12. However, the set

$S^* = \{u, n - u \mid u \text{ is a unitary divisor of } n\}$ is a symmetric subset of the group $(Z_n, \oplus)$. Using this symmetric subset of $(Z_n, \oplus)$, the unitary divisor Cayley graph is defined as follows:

Definition 2.2: Let $n \geq 1$ be an integer and let $S^* = \{u, n - u \mid u \text{ is a unitary divisor of } n \text{ other than } n\}$. The **Unitary Divisor Cayley graph** $G(Z_n, U_n)$ is the graph, whose vertex set is $V = \{1, 2, \dots, n - 1, n\}$ and the edge set $E = \{(x, y) \mid x - y \in S^*, \text{ or, } y - x \in S^*\}$.

For $n = 4, 6, 8, 12$ the unitary divisor Cayley graphs are given below:



Fig: $G(Z_4, U_4)$

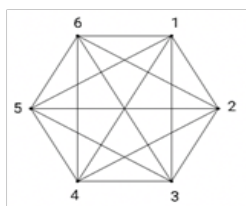


Fig: $G(Z_6, U_6)$



Fig: $G(Z_8, U_8)$

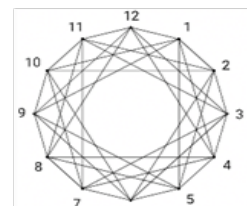


Fig: $G(Z_{12}, U_{12})$

- (i) If $n = 2k$, where k is odd, then $|S^*|$ is odd.
- (ii) If $n = 2^r q$, where $r > 1$ and q is odd, then $|S^*|$ is even.
- (iii) If n is odd, then $|S^*|$ is even.
- (iv) The graph $G(Z_n, U_n)$ is $|S^*|$ -regular. Moreover, the number of edges in $G(Z_n, U_n)$

$$\text{is } \frac{n |S^*|}{2}$$

- (v) The graph $G(Z_n, U_n)$ is connected.
- (vi) The graph $G(Z_n, U_n)$ is Hamiltonian
- (vii) The cycle $(1, 2, 3, \dots, (n - 1), n, 1)$ is called the outer Hamilton Cycle of the graph $G(Z_n, U_n)$.

3. Main Results

Lemma 3.1:

Let u be a unitary divisor of n . Then the graph $G(Z_n, U_n)$ contains exactly ' u ' disjoint cycles, each

- (c) Repeat the steps (a) and (b) recursively until we obtain the Hamilton cycle $H_{(0,1,\dots,u_1)}$ and spanning subgraph $C_{(0,1,\dots,u_1)}$ in which the vertices $0, 1, 2, \dots, u_1$ lie in the same cycle $C_{(0,1,\dots,u_1)}$ which is a Hamilton cycle.

The construction of the Hamilton cycles $H_{(0,1,\dots,u_1)}$ and $C_{(0,1,\dots,u_1)}$ shows that they are edge disjoint. Let us take $H_1 = H_{(0,1,\dots,u_1)}$ and $C_1 = C_{(0,1,\dots,u_1)}$. Now it is important to note that which of the edges of H , (that is, the outer edges) are transmitted to C_1 .

It is evident that the transmitted outer edges of C_1 are:

- i. $(0,1), (1,2), (2,3), \dots, (u_1-1, u_1)$.
- ii. $(u_1, u_1+1), (u_1+2, u_1+3), \dots, (2u_1-1, 2u_1)$ and
- iii. $(n-u_1+1, n-u_1+2), (n-u_1+3, n-u_1+4), \dots, (n-2, n-1)$

Let us take C_1 in place of H and the spanning subgraph consisting of all cycles generated by the next highest unitary divisor u_2 in the compliment C_1^c of C_1 in place of C . Since n is odd, u_1 and u_2 are also odd. So $u_2 < u_1$. Hence the set of edges $\{(0,1), \dots, (u_2-1, u_2)\}$ belong to the set of edges given in (i).

Moreover, $u_1, u_1+2, 2u_1-1, n-u_1+1, n-u_1+3, \dots$, are all odd numbers and $u_2 < u_1$, implies that the set of edges $\{(u_2, u_2+1), (u_2+2, u_2+3), \dots, (2u_2-1, 2u_2), (n-u_2+1, n-u_2+2), \dots, (n-2, n-1)\}$ belong to the set of all edges in (ii) and (iii).

Now repeat the steps (a) and (b) recursively with u_2 in place of u_1 up to the edge (u_2-1, u_2) . To these vertices we now adjoin edges of the type (i, j) for which $i-j = u_1$, till we obtain the new Hamilton cycle $H_{(0,1,\dots,u_2)}$ and the new spanning subgraph $C_{(0,1,\dots,u_2)}$.

Let us take $H_{(0,1,\dots,u_2)} = H_2$ and $C_{(0,1,\dots,u_2)} = C_2$. From the construction of H_1, H_2, C_2 it is clear that they are mutually edge disjoint.

Repeat the same procedure for the unitary divisors $u_3, u_4, \dots, u_m$, and obtain edge disjoint Hamilton Cycles $H_3, H_4, \dots, H_m$ and $C_m = H_{m+1}$ (say).

Thus corresponding to each unitary divisor $u_i, 0 \leq i \leq m$ of n , with $u_0 = n$, we obtain a Hamilton cycle H_{i+1} such that H_i 's are edge disjoint.

Further each H_i being Hamiltonian, contains exactly n edges. So, the number of edges in $H_1 \cup H_2 \cup \dots \cup H_{m+1}$ is given by $n(m+1)$. Since $G(Z_n, U_n)$ is $|S^*|$ -regular (by lemma 4.1.1) and

$$S^* = \{u_1, u_2, \dots, u_m, 1, n-u_1, n-u_2, \dots, n-u_m, n-1\}, \text{ we have}$$

$$|S^*| = 2(m+1) \quad (1)$$

So, the number of edges $\varepsilon(G)$ of $G(Z_n, U_n)$ is given by

$$\varepsilon(G) = n \frac{|S^*|}{2} = n(m+1) \quad (\text{by (1)})$$

$$= \text{the number of edges in } H_1 \cup H_2 \cup \dots \cup H_{m+1}$$

Hence the Theorem is proved.

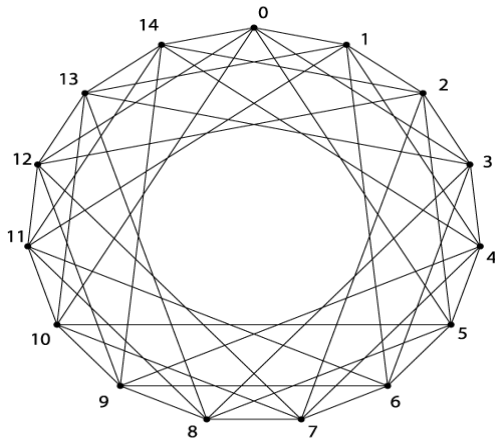
Corollary 3.5:

If n is odd, the number of edge disjoint Hamilton Cycles of $G(Z_n, U_n)$ is $u(n) - 1$, where $u(n)$ is the number of unitary divisors of n .

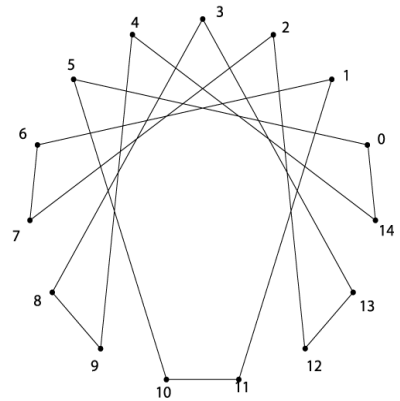
Proof:

The corollary is immediate from the above Theorem.

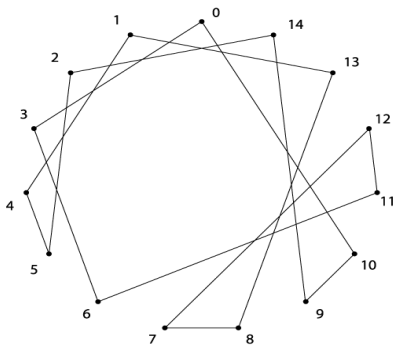
The three edge disjoint Hamilton cycles H_1, H_2 and H_3 of $G(Z_{15}, U_{15})$ are given below.



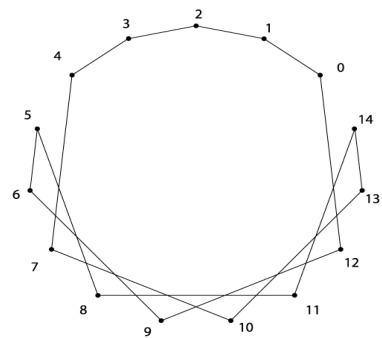
The Unitary divisor Cayley graph- $G (Z_{15}, U_{15})$



The Hamilton cycle H_1 of $G (Z_{15}, U_{15})$



The Hamilton cycle H_2 of $G (Z_{15}, U_{15})$



The Hamilton cycle H_3 of $G (Z_{15}, U_{15})$

Theorem 3.6:

Let $n = 2^r q$, where $r > 1$ and q is odd. If $u = 2^r t$, where t is odd is an even unitary divisor of n , then the unitary divisor Cayley graph $G (Z_n, U_n)$ can be decomposed into $2^r t$ edge disjoint cycles of length $\frac{n}{2^r t}$, which are not Hamilton cycles.

Proof:

Let $n = 2^r q$, where $r > 1$ and q is odd. Let $u = 2^r t$, where t is odd, be an even unitary divisor of n . Then $s = \frac{n}{2^r t}$ is odd and $(2^r t, \frac{n}{2^r t}) = 1$, since $2^r t$ is a unitary divisor of n . So $\frac{n}{2^r t}$ is also a unitary divisor of n and this is odd. Thus, corresponding to each even unitary divisor $u = 2^r t$ of n , then there is an odd unitary divisor $\frac{n}{2^r t}$ of n . Therefore the number of even unitary divisors of n is equal to the number of odd unitary divisors of n .

Let $u = 2^r t$ be an even unitary divisor of n . Consider the following sets of vertices in $G (Z_n, U_n)$.

$$C_1: (0 \quad 2^r t \quad 2 \cdot 2^r t \quad . \quad . \quad . \quad \frac{n}{2^r t} \cdot 2^r t \quad = 0)$$

$$C_2: (1 \quad 2^r t + 1 \quad 2 \cdot 2^r t + 1 \quad . \quad . \quad . \quad \frac{n}{2^r t} \cdot 2^r t + 1 \quad = 1)$$

$$C_3: (2^{2^r t + 2} \quad 2 \cdot 2^{2^r t + 2} \quad \dots \quad \frac{n}{2^r t} \cdot 2^{2^r t + 2} = 2)$$

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

$$C_{u=2^r t}: (2^{2^r t - 1} \quad 2^{2^r t} + (2^{2^r t} - 1) \quad 2 \cdot 2^{2^r t} + (2^{2^r t} - 1) \quad \dots \quad \frac{n}{2^r t} \cdot 2^{2^r t} + (2^{2^r t} - 1) = 2^{2^r t} - 1)$$

For any, k , $0 \leq i \leq 2^{2^r t} - 1$ and $0 \leq k \leq \frac{n}{2^r t}$, $[(k+1)2^{2^r t} + i] - [k \cdot 2^{2^r t} + i] = 2^{2^r t} \in S^*$, so that there is an edge

between these two vertices. Hence each C_i is a cycle of length $\frac{n}{2^r t}$. Further no two cycles have a vertex in common, so that they are disjoint cycles.

Let l be any vertex in $G(Z_n, U_n)$. By division algorithm applied to l and $2^{2^r t}$ we have $l = k \cdot 2^{2^r t} + i$, for some positive integers k and i such that $0 \leq k \leq \frac{n}{2^r t}$ and $0 \leq i \leq 2^{2^r t}$.

So, $l \in C_i$. This shows that $C_1 \cup C_2 \cup \dots \cup C_u = V$, the vertex set of G .

Hence corresponding to each even unitary divisor $u = 2^{2^r t}$ of n there are $2^{2^r t}$ edge disjoint cycles, each of length $\frac{n}{2^r t}$ in $G(Z_n, U_n)$.

Since $u = 2^{2^r t} > 1$, the length of each of the above cycle, namely, $\frac{n}{2^r t} < n$. So, none of these cycles is a Hamilton cycle.

Remark 3.7:

The Theorem 2.6 shows that corresponding to each even unitary divisor we cannot obtain a Hamilton Cycle disjoint from the outer Hamilton Cycle.

Theorem 3.8:

Let $n = 2^r q$, where $r > 1$ and q is odd. If $n > u_1 > u_2 > \dots > u_k > 1$ are odd unitary divisors of n , then the graph can be decomposed into $k+1$ edge disjoint Hamilton cycle.

Proof:

We have seen in the Theorem 4.2.6 that corresponding to each even unitary divisor, the edge set is decomposed into disjoint union of cycles, which are not Hamiltonian. So, first we delete (retaining the vertices) all the edges occurring in the edge disjoint cycles into which the edge set of G is decomposed corresponding to each of even unitary divisor and obtain the edge set E_1 .

Now consider the graph $G_1 = (V, E_1)$. Clearly, the graph G_1 contains the outer Hamilton cycle.

Now, let $n > u_1 > u_2 > \dots > u_k > 1$ be the odd unitary divisors. Applying the procedure adopted in the Theorem 4.2.4, one can see that G_1 can be decomposed into $k+1$ edge disjoint Hamilton cycles.

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